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# The Background Field Method for N = 2 Super Yang-Mills Theories in Harmonic Superspace

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**Abstract**

The background field method for $N = 2$ super Yang-Mills theories in harmonic superspace is developed. The ghost structure of the theory is investigated. It is shown that the ghosts include two fermionic real $\omega$-hypermultiplets (Faddeev-Popov ghosts) and one bosonic real $\omega$-hypermultiplet (Nielsen-Kallosh ghost), all in the adjoint representation of the gauge group. The one-loop effective action is analysed in detail and it is found that its structure is determined only by the ghost corrections in the pure super Yang-Mills theory. As applied to the case of $N = 4$ super Yang-Mills theory, realized in terms of $N = 2$ superfields, the latter result leads to the remarkable conclusion that the one-loop effective action of the theory does not contain quantum corrections depending on the $N = 2$ gauge superfield only. We show that the leading low-energy contribution to the one-loop effective action in the $N = 2$ $SU(2)$ super Yang-Mills theory coincides with Seiberg's perturbative holomorphic effective action.

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The background field method is a powerful and convenient tool for studying the structure of quantum gauge theories. Its main idea is based on the so-called background-quantum splitting of the initial gauge fields into two parts: the background fields and the quantum fields. To quantize the theory, one imposes the gauge fixing conditions only on the quantum fields, introduces the corresponding ghosts and considers the background fields as the functional arguments of the effective action. The gauge fixing functions are chosen to be background field dependent. As a result, we can find in concrete gauge models a class of gauge fixing functions with the property that the effective action will be invariant under the initial gauge transformations. The background field method was originally suggested by De Witt [1, 2] and then developed, and applied to concrete theories, by a large number of authors. The attractive feature of the background field method is that it preserves the manifest gauge invariance at each step of the loop calculations in quantum gauge theories.

Formulation of the background field method in $N = 1$ super Yang-Mills theory has been given in ref. [3] and its applications and generalizations were developed in detail (see [4]–[7] and also [8]–[10]). It turned out that the background-quantum splitting in $N = 1$ superfield Yang-Mills theory and supergravity is a non-trivial procedure as compared with the conventional Yang-Mills and gravity theories.

Construction of the background field method in extended supersymmetric gauge theories faces a fundamental problem. The most natural and proper description of such theories should be formulated in terms of a suitable superspace and unconstrained superfields over it. Therefore, the first step to developing the background field method in extended supersymmetric theories is a solution of the problem of formulating these theories in terms of unconstrained superfields.

An approach to constructing the background field method for $N = 2$ super Yang-Mills theories in the standard $N = 2$ superspace has been developed in ref. [11]. Some applications of this approach were investigated in refs. [12]. However, in our opinion, the approach of these authors looks very complicated from the technical point of view and its use for concrete problems should lead to a number of computational obstacles.

Interest in the quantum aspects of $N = 2$ super Yang-Mills theories has recently been revived by the seminal papers of Seiberg and Witten [13] (see [25] for a review), where the non-perturbative contribution to the low-energy effective action has been calculated. These calculations were based on the general structure of the low-energy effective action found in ref. [14] (see also [15]). The problem of the effective action in the $N = 2$ super Yang-Mills theory with matter has recently been studied in refs. [16]–[20]. However, all

these computations of the effective action in $N=2$ super Yang-Mills theories were given in terms of $N=1$ superfields without manifest realization of the $N=2$ supersymmetry.

The aim of this paper is to construct the background field method for $N=2$ super Yang-Mills theories and investigate the problem of the effective action in terms of unconstrained $N=2$ superfields[2]. We consider the formulation of $N=2$ super Yang-Mills theory in the harmonic superspace approach [21]–[24]. This approach provides a clear understanding of extended supersymmetric theories and opens opportunities to investigate both classical and quantum aspects of such theories. As we will see, the background field method formulation of $N=2$ super Yang-Mills theory in harmonic superspace is relatively simple. In particular, the structure of background-quantum splitting here is much more similar to the conventional Yang-Mills theory than to the $N=1$ super Yang-Mills case.

We start with a brief review of the pure $N=2$ super Yang-Mills (SYM) theory. In standard $N=2$ superspace with coordinates $z^M \equiv (x^m, \theta^\alpha_i, \bar\theta^i_{\dot\alpha})$, the gauge invariant action reads [26]

$$S_{\rm SYM} = \frac{1}{2g^2}{\rm tr}\int d^4x d^4\theta\, W^2 = \frac{1}{2g^2}{\rm tr}\int d^4x d^4\bar\theta\, \bar W^2 \tag{1}$$

where $W$ and $\bar W$ are the covariantly chiral superfield strength and its conjugate. These strengths are associated with the gauge covariant derivatives

$$\mathcal{D}_M \equiv (\mathcal{D}_m, \mathcal{D}^i_\alpha, \bar{\mathcal{D}}^{\dot\alpha}_i) = D_M + {\rm i}A_M \qquad A_M = A^a_M(z)T^a \tag{2}$$

satisfying the algebra [26]

$$\begin{gathered}
\{\mathcal{D}^i_\alpha, \bar{\mathcal{D}}_{\dot\alpha j}\} = -2{\rm i}\delta^i_j \mathcal{D}_{\alpha\dot\alpha} \\
\{\mathcal{D}^i_\alpha, \mathcal{D}^j_\beta\} = 2{\rm i}\varepsilon_{\alpha\beta}\varepsilon^{ij}\bar W \qquad \{\bar{\mathcal{D}}_{\dot\alpha i}, \bar{\mathcal{D}}_{\dot\beta j}\} = 2{\rm i}\varepsilon_{\dot\alpha\dot\beta}\varepsilon_{ij}W \\
[\mathcal{D}_{\alpha\dot\alpha}, \mathcal{D}^j_\beta] = \varepsilon_{\alpha\beta}\bar{\mathcal{D}}^i_{\dot\alpha}\bar W \qquad [\mathcal{D}_{\alpha\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta i}] = \varepsilon_{\dot\alpha\dot\beta}\mathcal{D}_{\alpha i}W\ .
\end{gathered} \tag{3}$$

Here $D_M \equiv (\partial_m, D^i_\alpha, \bar D^{\dot\alpha}_i)$ are the flat covariant derivatives, $T^a$ are the generators of the gauge group and ${\rm tr}\,(T^aT^b) = \delta^{ab}$ .

The covariant derivatives and a matter superfield multiplet $\varphi(z)$ transform as follows

$$\mathcal{D}'_M = e^{{\rm i}\tau}\mathcal{D}_M e^{-{\rm i}\tau} \qquad \varphi' = e^{{\rm i}\tau}\varphi \tag{4}$$

[2]We were informed by E. Ivanov that some aspects of the background field formulation for the $N=2$ super Yang-Mills theories were considered by A. Galperin, E. Ivanov and E. Sokatchev in unpublished work (private communication).

under the gauge group. Here $\tau = \tau^a(z)T^a$ and $\tau^a = \bar{\tau}^a$ are unconstrained real parameters. The set of all transformations (4) is said to form the $\tau$-group.

To realize the $N = 2$ SYM theory as a theory of unconstrained dynamical superfields, we extend the original superspace coordinates by bosonic ones $||u_i{}^{\mp}|| \in SU(2)$. These bosonic coordinates parametrize the two-sphere $SU(2)/U(1)$ and extend the superspace to $N = 2$ harmonic superspace [21]. Introducing the harmonic derivatives [21]

$$\begin{aligned} D^{\pm\pm} &= u^{\pm i}\frac{\partial}{\partial u^{\mp i}} \qquad && D^0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}} \\ [D^0, D^{\pm\pm}] &= \pm 2D^{\pm\pm} \qquad && [D^{++}, D^{--}] = D^0 \end{aligned} \tag{5}$$

and defining

$$\mathcal{D}_{\underline{M}} \equiv (\mathcal{D}_M, \mathcal{D}^{++}, \mathcal{D}^{--}, \mathcal{D}^0) \qquad \mathcal{D}^{\pm\pm} = D^{\pm\pm} \qquad \mathcal{D}^0 = D^0 \tag{6}$$

one observes that the operators $\mathcal{D}_{\underline{M}}$ possess the same transformation law (4) with respect to the $\tau$-group as $\mathcal{D}_M$. If we now introduce

$$\mathcal{D}^{\pm}_{\alpha} = u^{\pm}_i \mathcal{D}^i_{\alpha} \qquad \bar{\mathcal{D}}^{\pm}_{\dot{\alpha}} = u^{\pm}_i \bar{\mathcal{D}}^i_{\dot{\alpha}} \tag{7}$$

then the algebra of covariant derivatives takes the form

$$\begin{gathered} \{\mathcal{D}^+_\alpha, \mathcal{D}^+_\beta\} = \{\bar{\mathcal{D}}^+_{\dot\alpha}, \bar{\mathcal{D}}^+_{\dot\beta}\} = \{\mathcal{D}^+_\alpha, \bar{\mathcal{D}}^+_{\dot\alpha}\} = 0 \\ \{\mathcal{D}^+_\alpha, \mathcal{D}^-_\beta\} = -2\mathrm{i}\varepsilon_{\alpha\beta}\bar{W} \qquad \{\bar{\mathcal{D}}^+_{\dot\alpha}, \bar{\mathcal{D}}^-_{\dot\beta}\} = 2\mathrm{i}\varepsilon_{\dot\alpha\dot\beta}W \\ \{\bar{\mathcal{D}}^+_{\dot\alpha}, \mathcal{D}^-_\alpha\} = -\{\mathcal{D}^+_\alpha, \bar{\mathcal{D}}^-_{\dot\alpha}\} = 2\mathrm{i}\mathcal{D}_{\alpha\dot\alpha} \\ [\mathcal{D}^{++}, \mathcal{D}^+_\alpha] = [\mathcal{D}^{++}\bar{\mathcal{D}}^+_{\dot\alpha}] = 0 \\ [\mathcal{D}^{++}, \mathcal{D}^-_\alpha] = \mathcal{D}^+_\alpha \qquad [\mathcal{D}^{++}, \bar{\mathcal{D}}^-_{\dot\alpha}] = \bar{\mathcal{D}}^+_{\dot\alpha}\ . \end{gathered} \tag{8}$$

The relations in the first line imply

$$\mathcal{D}^+_\alpha = e^{-\mathrm{i}\Omega} D^+_\alpha e^{\mathrm{i}\Omega} \qquad \bar{\mathcal{D}}^+_{\dot\alpha} = e^{-\mathrm{i}\Omega} \bar{D}^+_{\dot\alpha} e^{\mathrm{i}\Omega} \tag{9}$$

for some Lie-algebra valued superfield $\Omega = \Omega^a(z,u)T^a$ with zero $U(1)$-charge, $D^0\Omega^a = 0$, and real, $\breve{\Omega}^{\,a} = \Omega^a$, with respect to the analyticity-preserving conjugation [21] which we denote here by $\smile$. They allow one to define covariantly analytic superfields constrained by

$$\mathcal{D}^+_\alpha \Phi^{(q)} = \bar{\mathcal{D}}^+_{\dot\alpha}\Phi^{(q)} = 0\ . \tag{10}$$

Here $\Phi^{(q)}(z,u)$ carries $U(1)$-charge $q$, $D^0\Phi^{(q)} = q\Phi^{(q)}$, and can be represented as follows

$$\Phi^{(q)} = e^{-\mathrm{i}\Omega}\phi^{(q)} \qquad D^+_\alpha\phi^{(q)} = \bar{D}^+_{\dot\alpha}\phi^{(q)} = 0 \tag{11}$$

with $\phi^{(q)}(\zeta_A, u)$ being an unconstrained superfield over an analytic subspace of the harmonic superspace [21] parametrized by $\zeta_A \equiv \{x^m_A, \theta^{+\alpha}, \bar\theta^+_{\dot\alpha}\}$ and $u^\pm_i$, where $x^m_A = x^m - 2\mathrm{i}\theta^{(i}\sigma^m\bar\theta^{j)}u^+_i u^-_j$ and $\theta^\pm_\alpha = u^\pm_i\theta^i_\alpha$, $\bar\theta^\pm_{\dot\alpha} = u^\pm_i\bar\theta^i_{\dot\alpha}$.

The $\Omega$ possesses a richer gauge freedom than the original $\tau$-group. Its transformation law reads

$$e^{\mathrm{i}\Omega'} = e^{\mathrm{i}\lambda}e^{\mathrm{i}\Omega}e^{-\mathrm{i}\tau} \tag{12}$$

with an unconstrained analytic gauge parameter $\lambda = \lambda^a(\zeta_A, u)T^a$ being real with respect to the analyticity-preserving conjugation, $\breve{\lambda}^{\,a} = \lambda^a$. The set of all $\lambda$-transformations form the so-called $\lambda$-group [21]. The $\tau$-group acts on $\Phi^{(q)}$ and leaves $\phi^{(q)}$ unchanged while the $\lambda$-group acts only on $\varphi^{(q)}$ as follows

$$\phi'^{(q)} = e^{\mathrm{i}\lambda}\phi^{(q)}\ . \tag{13}$$

The superfields $\Phi^{(q)}$ and $\varphi^{(q)}$ are said to correspond to $\tau$- and $\lambda$-frames respectively.

In the $\lambda$-frame, the covariant derivatives look like

$$\nabla_{\underline{M}} = e^{\mathrm{i}\Omega}\mathcal{D}_{\underline{M}}e^{-\mathrm{i}\Omega} \tag{14}$$

In particular

$$\begin{aligned} &\nabla^+_\alpha = D^+_\alpha \qquad \bar\nabla^+_{\dot\alpha} = \bar D^+_{\dot\alpha} \qquad \nabla^0 = D^0 \\ &\nabla^{\pm\pm} = e^{\mathrm{i}\Omega}D^{\pm\pm}e^{-\mathrm{i}\Omega} = D^{\pm\pm} + \mathrm{i}V^{\pm\pm}\ . \end{aligned} \tag{15}$$

In accordance with (8), the connection $V^{++} = V^{++a}T^a$ is a real analytic superfield, $\breve{V}^{\,++a} = V^{++a}$, $D^+_\alpha V^{++} = \bar D^+_{\dot\alpha}V^{++} = 0$, and its transformation law is

$$V'^{++} = e^{\mathrm{i}\lambda}V^{++}e^{-\mathrm{i}\lambda} - ie^{\mathrm{i}\lambda}D^{++}e^{-\mathrm{i}\lambda}\ . \tag{16}$$

The analytic superfield $V^{++}$ turns out to be the single unconstrained prepotential of the pure $N = 2$ SYM theory and all other objects are expressed in terms of it. In particular, action (1) can be rewritten via $V^{++}$ as follows [24]

$$S_{\mathrm{SYM}} = \frac{1}{g^2}\mathrm{tr}\int d^{12}z\sum_{n=2}^{\infty}\frac{(-\mathrm{i})^n}{n}\int du_1 du_2\ldots du_n\frac{V^{++}(z,u_1)V^{++}(z,u_2)\ldots V^{++}(z,u_n)}{(u^+_1u^+_2)(u^+_2u^+_3)\ldots(u^+_nu^+_1)}\ . \tag{17}$$

The rules of integration over $SU(2)$ as well as the properties of harmonic distributions are given in refs. [21, 22].

To quantize the theory under consideration we split $V^{++}$ into *background* $V^{++}$ and *quantum* $v^{++}$ parts

$$V^{++} \to V^{++} + gv^{++}\ . \tag{18}$$

Then, the original infinitesimal gauge transformations (16) can be realized in two different ways:

(i) *background transformations*

$$\delta V^{++} = -D^{++}\lambda - \mathrm{i}[V^{++},\lambda] = -\nabla^{++}\lambda \qquad \delta v^{++} = \mathrm{i}[\lambda, v^{++}] \tag{19}$$

(ii) *quantum transformations*

$$\delta V^{++} = 0 \qquad \delta v^{++} = -\frac{1}{g}\nabla^{++}\lambda - \mathrm{i}[v^{++},\lambda]\ . \tag{20}$$

It is worth pointing out that the form of the background-quantum splitting (18) and the corresponding background and quantum transformations (19), (20) are much more analogous to the conventional Yang-Mills theory than to the $N = 1$ non-abelian SYM model. Our aim now is to construct an effective action as a gauge-invariant functional of the background superfield $V^{++}$.

Upon the splitting (18), the classical action (17) can be shown to be given by

$$\begin{aligned} S_{\rm SYM}[V^{++} + gv^{++}] &= S_{\rm SYM}[V^{++}] + \frac{1}{4g}\mathrm{tr}\int d\zeta^{(-4)} du\, v^{++}\bar{D}^+_{\dot\alpha}\bar{D}^{+\dot\alpha}\bar{W}_\lambda \\ &+ \Delta S_{\rm SYM}[v^{++}, V^{++}]\ . \end{aligned} \tag{21}$$

Here $d\zeta^{(-4)} = d^4x_A d^2\theta^+ d^2\bar\theta^+$ and

$$\begin{aligned} \Delta S_{\rm SYM}[v^{++}, V^{++}] &= -\mathrm{tr}\int d^{12}z \sum_{n=2}^{\infty}\frac{(-ig)^{n-2}}{n}\int du_1 du_2\ldots du_n \\ &\times \frac{v^{++}_\tau(z,u_1)v^{++}_\tau(z,u_2)\ldots v^{++}_\tau(z,u_n)}{(u_1^+u_2^+)(u_2^+u_3^+)\ldots(u_n^+u_1^+)} \end{aligned} \tag{22}$$

$W_\lambda$, $\bar W_\lambda$ and $v^{++}_\tau$ denote the $\lambda$- and $\tau$-frame forms of $W$, $\bar W$ and $v^{++}$ respectively

$$\begin{aligned} W_\lambda = e^{\mathrm{i}\Omega}We^{-\mathrm{i}\Omega} \qquad \bar W_\lambda = e^{\mathrm{i}\Omega}\bar W e^{-\mathrm{i}\Omega} \\ v^{++}_\tau = e^{-\mathrm{i}\Omega}v^{++}e^{\mathrm{i}\Omega}\ . \end{aligned} \tag{23}$$

The superfield $\Omega$ corresponds to the background covariant derivatives constructed on the base of the background connection $V^{++}$. The quantum action $\Delta S_{\rm SYM}$ given in (22) depends on $V^{++}$ via the dependence of $v^{++}_\tau$ on $\Omega$, the latter being a complicated function of $V^{++}$. Each term in the action (21) is manifestly invariant with respect to the background

gauge transformations. The term linear in $v_\tau^{++}$ in (21) determines the equations of motion. This term should be dropped when considering the effective action[3].

To construct the effective action, we will follow the Faddeev-Popov Ansatz. Within the framework of the background field method, we should fix only the quantum transformations (20). Let us introduce the gauge fixing function in the form

$$\mathcal{F}_\tau^{(4)} = D^{++}v_\tau^{++} = e^{-\mathrm{i}\Omega}(\nabla^{++}v^{++})e^{\mathrm{i}\Omega} = e^{-\mathrm{i}\Omega}\mathcal{F}^{(4)}e^{\mathrm{i}\Omega} \tag{24}$$

which changes by the law

$$\delta\mathcal{F}_\tau^{(4)} = \frac{1}{g}e^{-\mathrm{i}\Omega}\{\nabla^{++}(\nabla^{++}\lambda + \mathrm{i}g[v^{++},\lambda])\}e^{\mathrm{i}\Omega} \tag{25}$$

under the quantum transformations (20). Eq. (25) leads to the Faddeev-Popov determinant

$$\Delta_{\mathrm{FP}}[v^{++},V^{++}] = \mathrm{Det}\,\nabla^{++}(\nabla^{++} + igv^{++})\ . \tag{26}$$

To get a path-integral representation for $\Delta_{\mathrm{FP}}[v^{++},V^{++}]$, we introduce two real analytic fermionic ghosts $\mathbf{b}$ and $\mathbf{c}$, in the adjoint representation of the gauge group, and the corresponding ghost action

$$S_{\mathrm{FP}}[\mathbf{b},\mathbf{c},v^{++},V^{++}] = \mathrm{tr}\int d\zeta_A^{(-4)}du\,\mathbf{b}\nabla^{++}(\nabla^{++}\mathbf{c} + \mathrm{i}g\,[v^{++},\mathbf{c}])\ . \tag{27}$$

As a result, we arrive at the effective action $\Gamma_{\mathrm{SYM}}[V^{++}]$ in the form

$$e^{\mathrm{i}\Gamma_{\mathrm{SYM}}[V^{++}]} = e^{\mathrm{i}S_{\mathrm{SYM}}[V^{++}]}\int\mathcal{D}v^{++}\mathcal{D}\mathbf{b}\mathcal{D}\mathbf{c}\,e^{\mathrm{i}(\Delta S_{\mathrm{SYM}}[v^{++},V^{++}]+S_{\mathrm{FP}}[\mathbf{b},\mathbf{c},v^{++},V^{++}])}\delta[\mathcal{F}^{(4)} - f^{(4)}] \tag{28}$$

where $f^{(4)}(\zeta_A,u)$ is an external Lie-algebra valued analytic superfield independent of $V^{++}$, and $\delta[\mathcal{F}^{(4)}]$ is the proper functional analytic delta-function.

To transform the path integral for $\Gamma_{\mathrm{SYM}}[V^{++}]$ to a more useful form, we average the right hand side in eq. (28) with the weight

$$\Delta[V^{++}]\ \exp\left\{\frac{\mathrm{i}}{2\alpha}\mathrm{tr}\int d^{12}zdu_1du_2\ f_\tau^{(4)}(z,u_1)\frac{(u_1^-u_2^-)}{(u_1^+u_2^+)^3}f_\tau^{(4)}(z,u_2)\right\} \tag{29}$$

Here $\alpha$ is an arbitrary (gauge) parameter. In flat superspace, a weight function of this form has been used in refs. [22, 23]. The functional $\Delta[V^{++}]$ should be chosen from the

[3] As is well known, for calculating the effective action within the loop expansion one really uses the construction $\Delta S[\Psi,\psi] = S[\Psi+\psi] - S[\Psi] - S'[\Psi]\psi$ , where the linear term is absent (see, f.e., [27]). Here $\Psi$ denotes the set of all fields of the theory and we split $\Psi \to \Psi + \psi$, with $\Psi$ the background field and $\psi$ the quantum one.

condition

$$1 = \Delta[V^{++}] \int \mathcal{D} f^{(4)} \exp\left\{ \frac{\mathrm{i}}{2\alpha} \mathrm{tr} \int d^{12}z du_1 du_2\, f^{(4)}_\tau(z,u_1) \frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3} f^{(4)}_\tau(z,u_2) \right\} \qquad (30)$$

hence

$$\begin{aligned} \Delta^{-1}[V^{++}] &= \int \mathcal{D} f^{(4)} \exp\left\{ \frac{\mathrm{i}}{2\alpha} \mathrm{tr} \int d\zeta_1^{(-4)} d\zeta_2^{(-4)} du_1 du_2\, f^{(4)}(\zeta_1,u_1) A(1,2) f^{(4)}(\zeta_2,u_2) \right\} \\ &= \mathrm{Det}^{-1/2} A \end{aligned} \qquad (31)$$

for a special background-dependent operator $A$ acting on the space of analytic superfields with values in the Lie algebra of the gauge group. Thus

$$\Delta[V^{++}] = \mathrm{Det}^{1/2} A\ . \qquad (32)$$

To find $\mathrm{Det}\, A$ we represent it by a functional integral over analytic superfields of the form

$$\mathrm{Det}^{-1} A = \int \mathcal{D}\chi^{(4)} \mathcal{D}\rho^{(4)} \exp\left\{ \mathrm{i}\ \mathrm{tr} \int d\zeta_1^{(-4)} du_1 d\zeta_2^{(-4)} du_2\, \chi^{(4)}(1) A(1,2) \rho^{(4)}(2) \right\} \qquad (33)$$

and perform the following replacement of functional variables

$$\rho^{(4)} = (\nabla^{++})^2 \sigma \qquad \mathrm{Det}\left( \frac{\delta \rho^{(4)}}{\delta \sigma} \right) = \mathrm{Det}\, (\nabla^{++})^2\ . \qquad (34)$$

Then we have

$$\begin{aligned} & \mathrm{tr} \int d\zeta_1^{(-4)} du_1 d\zeta_2^{(-4)} du_2\, \chi^{(4)}(1) A(1,2) \rho^{(4)}(2) \\ &= \mathrm{tr} \int d^{12}z du_1 du_2\, \chi^{(4)}_\tau \frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3} (D_2^{++})^2 \sigma_\tau(2) = \frac{1}{2} \mathrm{tr} \int d^{12}z du\, \chi^{(4)}_\tau (D^{--})^2 \sigma_\tau \\ &= -\mathrm{tr} \int d\zeta^{(-4)} du\, \chi^{(4)}\, \widehat{\Box}\, \sigma \end{aligned} \qquad (35)$$

where[4]

$$\widehat{\Box} = -\frac{1}{2} (\nabla^+)^4 (\nabla^{--})^2 = -\frac{1}{2} (D^+)^4 (\nabla^{--})^2\ . \qquad (36)$$

On the basis of eqs. (32–35) one obtains

$$\Delta[V^{++}] = \mathrm{Det}^{-\frac{1}{2}} (\nabla^{++})^2 \mathrm{Det}^{\frac{1}{2}}\, \widehat{\Box}\ \ . \qquad (37)$$

[4] We use the notation $(D^+)^4 = \frac{1}{16}(D^+)^2(\bar{D}^+)^2$, $(D^\pm)^2 = D^{\pm\alpha} D^\pm_\alpha$, $(\bar{D}^\pm)^2 = \bar{D}^\pm_{\dot\alpha} \bar{D}^{\pm\dot\alpha}$ and similar notation for the gauge-covariant derivatives.

Below, it will be proven that

$$\mathrm{Det}\ \widehat{\Box} = 1\ . \tag{38}$$

Therefore, we are able to represent $\Delta[V^{++}]$ by the following functional integral

$$\begin{aligned}\Delta[V^{++}] &= \int \mathcal{D}\phi\, e^{\mathrm{i}S_{\mathrm{NK}}[\phi,V^{++}]} \\ S_{\mathrm{NK}}[\phi,V^{++}] &= -\frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)}du\,\nabla^{++}\phi\nabla^{++}\phi \end{aligned}\tag{39}$$

with the integration variable $\phi$ being a bosonic real analytic superfield taking its values in the Lie algebra of the gauge group. The $\phi$ is in fact the Nielsen-Kallosh ghost for the theory. As a result, we see that the $N=2$ SYM theory is described within the background field approach by three ghosts: the two fermionic ghosts $\mathbf{b}$ and $\mathbf{c}$ and the third bosonic ghost $\phi$. The ghost actions $S_{\mathrm{FP}}$ and $S_{\mathrm{NK}}$ given by eqs. (27) and (39) correspond to the known $\omega$-hypermultiplet [21].

Upon averaging the effective action $\Gamma_{\mathrm{SYM}}[V^{++}]$ with the weight (29), one gets the following path integral representation

$$e^{\mathrm{i}\Gamma_{\mathrm{SYM}}[V^{++}]} = e^{\mathrm{i}S_{\mathrm{SYM}}[V^{++}]}\int \mathcal{D}v^{++}\mathcal{D}\mathbf{b}\mathcal{D}\mathbf{c}\mathcal{D}\phi\, e^{\mathrm{i}S_{\mathrm{Q}}[v^{++},\mathbf{b},\mathbf{c},\phi,V^{++}]} \tag{40}$$

where

$$\begin{aligned}S_{\mathrm{Q}}[v^{++},\mathbf{b},\mathbf{c},\phi,V^{++}] &= \Delta S_{\mathrm{SYM}}[v^{++},V^{++}] + S_{\mathrm{GF}}[v^{++},V^{++}] \\ &+ S_{\mathrm{FP}}[\mathbf{b},\mathbf{c},v^{++},V^{++}] + S_{\mathrm{NK}}[\phi,V^{++}]\ .\end{aligned}\tag{41}$$

Here $S_{\mathrm{GF}}[v^{++},V^{++}]$ is the gauge fixing contribution to the quantum action

$$\begin{aligned}S_{\mathrm{GF}}[v^{++},V^{++}] &= \frac{1}{2\alpha}\mathrm{tr}\int d^{12}z du_1 du_2 \frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3}(D_1^{++}v_\tau^{++}(1))(D_2^{++}v_\tau^{++}(2)) \\ &= \frac{1}{2\alpha}\mathrm{tr}\int d^{12}z du_1 du_2\, \frac{v_\tau^{++}(1)v_\tau^{++}(2)}{(u_1^+ u_2^+)^2} - \frac{1}{4\alpha}\mathrm{tr}\int d^{12}z du\, v_\tau^{++}(D^{--})^2 v_\tau^{++}\end{aligned}\tag{42}$$

Let us consider the sum of the quadratic part in $v^{++}$ of $\Delta S_{\mathrm{SYM}}$ (22) and $S_{\mathrm{GF}}$ (42). It has the form

$$\frac{1}{2}(1+\frac{1}{\alpha})\mathrm{tr}\int d^{12}z du_1 du_2\, \frac{v_\tau^{++}(1)v_\tau^{++}(2)}{(u_1^+ u_2^+)^2} + \frac{1}{2\alpha}\mathrm{tr}\int d\zeta^{(-4)}du\, v^{++}\,\widehat{\Box}\,v^{++} \tag{43}$$

where we have used eq. (36). To further simplify the computation, we set $\alpha = -1$. We can now write the final result for the effective action $\Gamma_{\mathrm{SYM}}[V^{++}]$

$$e^{\mathrm{i}\Gamma_{\mathrm{SYM}}[V^{++}]} = e^{\mathrm{i}S_{\mathrm{SYM}}[V^{++}]}\int \mathcal{D}v^{++}\mathcal{D}\mathbf{b}\mathcal{D}\mathbf{c}\mathcal{D}\phi\, e^{\mathrm{i}S_{\mathrm{Q}}[v^{++},\mathbf{b},\mathbf{c},\phi,V^{++}]} \tag{44}$$

where action $S_{\rm Q}$ is as follows

$$S_{\rm Q}[v^{++},\mathbf{b},\mathbf{c},\phi,V^{++}] = S_2[v^{++},{\rm b},\mathbf{c},\phi,V^{++}] + S_{\rm int}[v^{++},\mathbf{b},\mathbf{c},V^{++}] \tag{45}$$

$$\begin{aligned} S_2[v^{++},\mathbf{b},\mathbf{c},\phi,V^{++}] = & -\frac{1}{2}{\rm tr}\int d\zeta^{(-4)}du\, v^{++}\,\widehat{\Box}\,v^{++} + {\rm tr}\int d\zeta^{(-4)}du\,\mathbf{b}(\nabla^{++})^2\mathbf{c} \\ & +\frac{1}{2}{\rm tr}\int d\zeta^{(-4)}du\,\phi(\nabla^{++})^2\phi \end{aligned} \tag{46}$$

$$\begin{aligned} S_{\rm int}[v^{++},\mathbf{b},{\rm c},V^{++}] = & -{\rm tr}\int d^{12}z du_1\ldots du_n \sum_{n=3}^{\infty}\frac{(-ig)^{n-2}}{n}\frac{v^{++}_\tau(z,u_1)\ldots v^{++}_\tau(z,u_n)}{(u^+_1u^+_2)\ldots(u^+_nu^+_1)} \\ & -{\rm i}g\,{\rm tr}\int d\zeta^{(-4)}du\nabla^{++}\mathbf{b}\,[v^{++},\mathbf{c}]\,. \end{aligned} \tag{47}$$

Eqs. (44–47) completely determine the structure of the perturbation expansion for calculating the effective action $\Gamma_{\rm SYM}[V^{++}]$ of the pure $N=2$ SYM theory in a manifestly supersymmetric and gauge invariant form.

Let us now prove the relation (38). We proceed by pointing out that $\widehat{\Box}$ transforms each covariantly analytic superfield into a covariantly analytic one. When acting on spaces of such superfields, $\widehat{\Box}$ is equivalent to the second-order differential operator

$$\begin{aligned} \widehat{\Box}_\tau = e^{-{\rm i}\Omega}\,\widehat{\Box}\,e^{{\rm i}\Omega} \;=\; & \mathcal{D}^m\mathcal{D}_m + \frac{\rm i}{2}(\mathcal{D}^{+\alpha}W)\mathcal{D}^-_\alpha + \frac{\rm i}{2}(\bar{\mathcal{D}}^+_{\dot\alpha}\bar W)\bar{\mathcal{D}}^{-\dot\alpha} - \frac{\rm i}{4}(\bar{\mathcal{D}}^+_{\dot\alpha}\bar{\mathcal{D}}^{+\dot\alpha}\bar W)\mathcal{D}^{--} \\ + \; & \frac{\rm i}{4}(\mathcal{D}^{-\alpha}\mathcal{D}^+_\alpha W) + \bar W W \end{aligned} \tag{48}$$

as a consequence of the covariant derivative algebra (8). It is remarkable that the differential part of $\widehat{\Box}$ is uniquely determined from the requirements that (i) $\widehat{\Box}$ is constructed in terms of the covariant derivatives only; (ii) $\widehat{\Box}$ moves every covariantly analytic superfield into a covariantly analytic one; (iii) $\widehat{\Box}$ is a second-order operator containing the only term $\mathcal{D}^m\mathcal{D}_m$ with two vector covariant derivatives. Next, we introduce the proper-time representation for a regularized form of Det $\widehat{\Box}$ (see ref. [10] for more details of the superfield proper-time technique)

$$\ln\,({\rm Det}\;\widehat{\Box})_{\rm reg} = -\mu^{2\varepsilon}\int_0^\infty d({\rm i}s)({\rm i}s)^{\varepsilon-1}{\rm Tr}\,e^{{\rm i}s\widehat{\Box}}. \tag{49}$$

Here we have introduced the regularization parameters $\mu$ and $\varepsilon$, where $\varepsilon\to 0$ in the end of calculations. The Tr of the analytic 'heat kernel' $e^{{\rm i}s\widehat{\Box}}$ is defined by

$${\rm Tr}\,e^{{\rm i}s\widehat{\Box}} = {\rm tr}\int d\zeta_1^{(-4)}du_1 d\zeta_2^{(-4)}du_2\,\delta_A^{(2,2)}(\zeta_1,u_1|\zeta_2,u_2)e^{{\rm i}s\widehat{\Box}}\delta_A^{(2,2)}(\zeta_1,u_1|\zeta_2,u_2) \tag{50}$$

and $\delta_A^{(2,2)}(1,2)$ denotes the proper analytic subspace delta-function

$$\begin{aligned} \delta^{(2,2)}(\zeta_1,u_1|\zeta_2,u_2) &= (D_1^+)^4\{\delta^{12}(z_1-z_2)\delta^{(-2,2)}(u_1,u_2)\} \\ &= (D_2^+)^4\{\delta^{12}(z_1-z_2)\delta^{(2,-2)}(u_1,u_2)\} \end{aligned} \tag{51}$$

with $\delta^{(-2,2)}(u_1,u_2)$ and $\delta^{(2,-2)}(u_1,u_2)$ being special harmonic delta-functions [22]. Further, we rewrite

$$e^{\mathrm{i}s\widehat{\Box}} = 1 + \widehat{\Box}\ \frac{e^{\mathrm{i}s\widehat{\Box}}-1}{\widehat{\Box}} = 1 - \frac{1}{2}(D^+)^4(\nabla^{--})^2\mathcal{U}(s) \tag{52}$$

where

$$\mathcal{U}(s) = \frac{e^{\mathrm{i}s\widehat{\Box}}-1}{\widehat{\Box}}\ . \tag{53}$$

Then

$$\begin{aligned}\mathrm{Tr}\, e^{\mathrm{i}s\widehat{\Box}} &= -\frac{1}{2}\mathrm{tr}\int d\zeta_1^{(-4)}du_1 d\zeta_2^{(-4)}du_2\, \delta_A^{(2,2)}(\zeta_1,u_1|\zeta_2,u_2) \\ &\times\ (D_1^+)^4(\nabla_1^{--})^2\mathcal{U}(s)\delta_A^{(2,2)}(\zeta_1,u_1|\zeta_2,u_2)\ .\end{aligned} \tag{54}$$

Taking into account the explicit form of $\widehat{\Box}$ and making use of the covariant derivative algebra (8), one readily observes

$$\begin{aligned}\mathcal{U}_\tau(s) &= e^{-\mathrm{i}\Omega}\mathcal{U}(s)e^{\mathrm{i}\Omega} \\ &= A(s) + B^{+\alpha}(s)\mathcal{D}^-_\alpha + \tilde{B}^{+\dot\alpha}(s)\bar{\mathcal{D}}^-_{\dot\alpha} + C^{++}(s)(\mathcal{D}^-)^2 + \tilde{C}^{++}(s)(\bar{\mathcal{D}}^-)^2 \\ &+\ E^{++\alpha\dot\alpha}(s)[\mathcal{D}^-_\alpha,\bar{\mathcal{D}}^-_{\dot\alpha}] + F^{(3)\alpha}\mathcal{D}^-_\alpha(\bar{\mathcal{D}}^-)^2 + \tilde{F}^{(3)\dot\alpha}(s)\bar{\mathcal{D}}^-_{\dot\alpha}(\mathcal{D}^-)^2 \\ &+\ G^{(4)}(s)(\mathcal{D}^-)^4\ .\end{aligned} \tag{55}$$

Here $A$, $B^{+\alpha}$, $\tilde{B}^{+\dot\alpha}$, $C^{++}$, $\tilde{C}^{++}$, $E^{++\alpha\dot\alpha}$, $F^{(3)\alpha}$, $\tilde{F}^{(3)\dot\alpha}$ and $G^{(4)}$ are some functions of the real parameter $s$, the covariant derivatives $\mathcal{D}_m$, $\mathcal{D}^{--}$ as well as of $W$, $\bar{W}$ and their covariant derivatives. The exact form of these functions is not essential here. The dependence of $\mathcal{U}_\tau(s)$ on the spinor covariant derivatives has been written down in eq. (55) explicitly and this is all that we will need later on.

The integrals over the analytic subspace in eq. (54) can be transformed into integrals over the full superspace

$$\begin{aligned}\mathrm{Tr}\, e^{\mathrm{i}s\widehat{\Box}} &= -\frac{1}{2}\mathrm{tr}\int d^{12}z_1du_1d^{12}z_2du_2\delta^{12}(z_1-z_2)\delta^{(2,-2)}(u_1,u_2) \\ &\times\ (\nabla_1^{--})^2\mathcal{U}(s)\delta_A^{(2,2)}(\zeta_1,u_1|\zeta_2,u_2) \\ &= -\frac{1}{2}\mathrm{tr}\int d^{12}z_1du_1d^{12}z_2du_2\delta^{12}(z_1-z_2)\delta^{(2,-2)}(u_1,u_2) \\ &\times\ (D_1^{--})^2\mathcal{U}_\tau(s)(\mathcal{D}_1^+)^4\delta^{12}(z_1-z_2)\delta^{(-2,2)}(u_1,u_2)\ .\end{aligned} \tag{56}$$

Because of eq. (55), we can continue in the manner

$$\mathrm{Tr}\, e^{\mathrm{i}s\widehat{\Box}} = -\frac{1}{2}\mathrm{tr}\int d^{12}z_1du_1d^{12}z_2du_2\delta^{12}(z_1-z_2)\delta^{(2,-2)}(u_1,u_2)$$

$$
\begin{aligned}
&\times \quad (D_1^{--})^2 G^{(4)}(s)(\mathcal{D}_1^-)^4(\mathcal{D}_1^+)^4\delta^{12}(z_1-z_2)\delta^{(-2,2)}(u_1,u_2)\\
&= \quad -\frac{1}{2}\mathrm{tr}\int d^{12}z_1 du_1 d^{12}z_2 du_2[\delta^8(\theta_1-\theta_2)(D_1^-)^4(D_1^+)^4\delta^8(\theta_1-\theta_2)]\\
&\times \quad \delta^4(x_1-x_2)[(D_1^{--})^2\delta^{(2,-2)}(u_1,u_2)]G^{(4)}(s)\delta^{(-2,2)}(u_1,u_2)\delta^4(x_1-x_2)\,. \quad (57)
\end{aligned}
$$

Since

$$
\int d\theta_2^8\,\delta^8(\theta_1-\theta_2)(D_1^-)^4(D_1^+)^4\delta^8(\theta_1-\theta_2)=1
$$

we obtain

$$
\begin{aligned}
\mathrm{Tr}\, e^{\mathrm{i}s\widehat{\Box}} \quad &= \quad -\frac{1}{2}\mathrm{tr}\int d^8\theta d^4x_1 du_1 d^4x_2 du_2 \delta^4(x_1-x_2)\\
&\times \quad [(D_1^{--})^2\delta^{(2,-2)}(u_1,u_2)]G^{(4)}(s)\delta^{(-2,2)}(u_1,u_2)\delta^4(x_1-x_2)=0 \qquad (58)
\end{aligned}
$$

as a consequence of the following property of harmonic delta-functions

$$
\int du_1 du_2\,\delta^{(m,-m)}(u_1,u_2)\,f^{(\pm 2p)}(u_1)\,(D_1^{\mp\mp})^p\,\delta^{(-m,m)}(u_1,u_2)=0 \qquad p>0 \quad (59)
$$

$$
\int du_1 du_2\,\delta^{(0,0)}(u_1,u_2)\,f^{(0)}(u_1)\,\delta^{(0,0)}(u_1,u_2)=f^{(0)}(0)\,\infty \qquad p=0 \quad (60)
$$

with $f^{(\pm 2p)}(u)$ an arbitrary function of $U(1)$-charge $\pm 2p$. Eqs. (59) and (60) can be readily justified if one regularizes the harmonic delta-function [22]

$$
\delta^{(m,-m)}(u_1,u_2)=\sum_{n=0}^{\infty}(-1)^{n+m}\frac{(2n+m+1)!}{n!(n+m)!}(u_1^+)_{(n+m}(u_1^-)_{n)}(u_2^+)^{(n}(u_2^-)^{n+m)} \qquad (61)
$$

by cutting off the Fourier series at the upper limit

$$
\delta_N^{(m,-m)}(u_1,u_2)\equiv\sum_{n=0}^{N}(-1)^{n+m}\frac{(2n+m+1)!}{n!(n+m)!}(u_1^+)_{(n+m}(u_1^-)_{n)}(u_2^+)^{(n}(u_2^-)^{n+m)} \qquad (62)
$$

where $N\to\infty$ in the end of the calculation. Eq. (59) remains valid for the regularized harmonic delta-functions. Therefore, we have proven relation (38).

The generic expressions (44–47) open an opportunity to investigate the loop corrections to the effective action $\Gamma_{\rm SYM}[V^{++}]$. Let us consider the one-loop approximation. In this case the effective action has the structure $\Gamma_{\rm SYM}[V^{++}]=S_{\rm SYM}[V^{++}]+\Gamma^{(1)}_{\rm SYM}[V^{++}]$, where $\Gamma^{(1)}_{\rm SYM}[V^{++}]$ describes the one-loop quantum corrections. The relations (44) and (45) along with eq. (38) immediately lead to

$$
\Gamma^{(1)}_{\rm SYM}[V^{++}]=-\mathrm{i}\left(\mathrm{Tr}\ln(\nabla^{++})^2-\frac{1}{2}\mathrm{Tr}\ln(\nabla^{++})^2\right)=-\frac{\mathrm{i}}{2}\mathrm{Tr}\ln(\nabla^{++})^2\,. \qquad (63)
$$

It is remarkable that even if the relation (38) was not true, $\mathrm{Tr}\ln\widehat{\Box}$ would not enter $\Gamma^{(1)}[V^{++}]$ anyway. In such a case we should start from eq. (37), keeping $\mathrm{Det}\,\widehat{\Box}$ intact at all stages, and would obtain for $\Gamma^{(1)}[V^{++}]$ the following representation

$$\Gamma^{(1)}[V^{++}] = -\mathrm{i}\left(\frac{1}{2}\mathrm{Tr}\ln\widehat{\Box} - \frac{1}{2}\mathrm{Tr}\ln\widehat{\Box}\right) - \frac{\mathrm{i}}{2}\mathrm{Tr}\ln(\nabla^{++})^2 . \tag{64}$$

As a result, the whole contribution to the one-loop effective action is stipulated only by the ghost contribution. Moreover, this ghost contribution differs only in sign from the contribution of a single real $\omega$-hypermultiplet, in the adjoint representation of the gauge group, coupled to the external gauge superfield $V^{++}$. The structure of the effective action of the $\omega$-multiplet has been investigated in our previous paper [28] for an abelian gauge group, and that work is readily extended to the non-abelian case.

We have developed the background field method for the pure $N=2$ SYM theory. In the general case, the classical action contains not only the pure SYM part given by (1) (or, what is equvalent, by (17)), but also the matter action of the general form [21]

$$S_{\mathrm{MAT}} = -\int d\zeta^{(-4)}du\,\breve{q}^{\;+}\nabla^{++}q^{+} - \frac{1}{2}\int d\zeta^{(-4)}du\,\nabla^{++}\omega^{\mathrm{T}}\nabla^{++}\omega \tag{65}$$

describing the matter $q$-hypermultiplet $(q^{+}(\zeta_A,u), \breve{q}^{\;+}(\zeta_A,u))$ and $\omega$-hypermultiplet $\omega(\zeta_A,u)$ coupled to the SYM gauge superfield $V^{++}$. Our previous considerations can be easily extended to the case of the general $N=2$ SYM theory. The only non-trivial new information, however, is the explicit structure of the matter superpropagators associated with the above action (65). They read as follows

$$\begin{aligned} G^{(1,1)}_{\mathrm{F}}(1,2) &\equiv \mathrm{i} < q^{+}(1)\;\breve{q}^{\;+}(2) > \\ &= -\frac{1}{\widehat{\Box}_{1}}(D^{+}_{1})^{4}(D^{+}_{2})^{4}\left\{\delta^{4}(x_1-x_2)\delta^{8}(\theta_1-\theta_2)\frac{1}{(u^{+}_{1}u^{+}_{2})^{3}}e^{\mathrm{i}\Omega(1)}e^{-\mathrm{i}\Omega(2)}\right\} \quad (66) \\ G^{(0,0)}_{\mathrm{F}}(1,2) &\equiv \mathrm{i} < \omega(1)\,\omega^{\mathrm{T}}(2) > \\ &= -\frac{1}{\widehat{\Box}_{1}}(D^{+}_{1})^{4}(D^{+}_{2})^{4}\left\{\delta^{4}(x_1-x_2)\delta^{8}(\theta_1-\theta_2)\frac{(u^{-}_{1}u^{-}_{2})}{(u^{+}_{1}u^{+}_{2})^{3}}e^{\mathrm{i}\Omega(1)}e^{-\mathrm{i}\Omega(2)}\right\} \quad (67) \end{aligned}$$

and satisfy the equations

$$\nabla^{++}_{1}G^{(1,1)}_{\mathrm{F}}(1,2) = \delta^{(3,1)}_{A}(1,2) \tag{68}$$

$$(\nabla^{++}_{1})^{2}G^{(0,0)}_{\mathrm{F}}(1,2) = -\delta^{(4,0)}_{A}(1,2) \tag{69}$$

respectively, with $\widehat{\Box}$ given by (48). Switching off the gauge superfield, the Green's functions turn into the free ones obtained in [22]. The Green's functions (66) and (67) are to be used for loop calculations in the background field approach.

As the simplest application of the techniques developed here, we demonstrate the fact that the one-loop quantum correction to the effective action of the $N=4$ SYM theory realized in terms of $N=2$ superfields does not contain contributions depending only on the $N=2$ gauge superfield. In $N=2$ superspace, this theory is described by the action

$$S^{N=4}_{\rm SYM} = \frac{1}{2g^2}{\rm tr}\int d^4x d^4\theta\, W^2 \; - \; \frac{1}{2g^2}\,{\rm tr}\int d\zeta^{(-4)} du\, \nabla^{++}\omega\nabla^{++}\omega \tag{70}$$

with $\omega$ the real $\omega$-hypermultiplet taking its values in the Lie algebra of the gauge group. This action was shown to possess $N=4$ supersymmetry [23] transforming $V^{++}$ and $\omega$ into each other. We denote by $\Gamma[V^{++},\omega]$ the effective action of the theory and consider the one-loop correction $\Gamma^{(1)}[V^{++},\omega]$. The contributions to $\Gamma^{(1)}[V^{++},\omega]$, which depend only on $V^{++}$, come from (63) as well from the matter functional integral

$$e^{{\rm i}\Gamma^{(1)}_{\rm MAT}[V^{++}]} = \int \mathcal{D}\omega e^{-{\rm i}\frac{1}{2g^2}\,{\rm tr}\int d\zeta^{(-4)}du\,\nabla^{++}\omega\nabla^{++}\omega} \qquad \Gamma^{(1)}_{\rm MAT}[V^{++}] = \frac{\rm i}{2}{\rm Tr}\ln(\nabla^{++})^2\,. \tag{71}$$

But $\Gamma^{(1)}_{\rm SYM}[V^{++}]$ and $\Gamma^{(1)}_{\rm MAT}[V^{++}]$ exactly cancel each other.

Finally, we would like to discuss the leading low-energy contribution to the one-loop effective action in the $N=2$ $SU(2)$ SYM theory with the gauge group spontaneously broken to $U(1)$. Here the one-loop effective $\Gamma^{(1)}_{SU(2)}[V^{++}]$ reads

$$\Gamma^{(1)}_{SU(2)}[V^{++}] = -\Gamma_{\phi}[V^{++}] \tag{72}$$

with $\Gamma_\phi[V^{++}]$ the effective action of a real $\omega$-hypermultiplet in the adjoint representation of $SU(2)$ coupled to the external gauge superfield $V^{++}$ :

$$e^{{\rm i}\,\Gamma_\phi[V^{++}]} = \int \mathcal{D}\phi\, \exp\left\{-\frac{\rm i}{2}\,{\rm tr}\int d\zeta^{(-4)} du \nabla^{++}\phi\,\nabla^{++}\phi\right\} \tag{73}$$

where

$$\begin{gathered}\phi = \phi^a\tau^a \qquad \nabla^{++}\phi = D^{++}\phi + {\rm i}[V^{++},\phi] \\ \tau^a = \tfrac{1}{\sqrt{2}}\sigma^a \qquad [\tau^a,\tau^b] = {\rm i}\sqrt{2}\varepsilon^{abc}\tau^c \qquad {\rm tr}\,(\tau^a\tau^b) = \delta^{ab}\;.\end{gathered} \tag{74}$$

Upon the spontaneous breakdown of $SU(2)$, only the $U(1)$ gauge symmetry survives and the gauge superfield $V^{++} = V^{++a}\tau^a$ takes the form

$$V^{++} = V^{++3}\tau^3 \equiv \mathcal{V}^{++}\tau^3\;. \tag{75}$$

Here $\mathcal{V}^{++}$ consists of two parts, $\mathcal{V}^{++} = \mathcal{V}^{++}_0 + \mathcal{V}^{++}_1$, where $\mathcal{V}^{++}_0$ corresponds to a constant strength $\mathcal{W}_0 = const$, and $\mathcal{V}^{++}_1$ is an abelian gauge superfield. It can be proved that the

presence of $\mathcal{V}_0^{++}$ leads to the appearance of mass $|\mathcal{W}_0|^2$ for matter multiplets (see [28]). Now, we have

$$
\begin{aligned}
\nabla^{++}\phi^1 &= D^{++}\phi^1 + \sqrt{2}\mathcal{V}^{++}\phi^2 \\
\nabla^{++}\phi^2 &= D^{++}\phi^2 - \sqrt{2}\mathcal{V}^{++}\phi^1 \\
\nabla^{++}\phi^3 &= D^{++}\phi^3 \,.
\end{aligned} \tag{76}
$$

Thus $\phi^3$ completely decouples. Unifying $\phi^1$ and $\phi^2$ in to the *complex* $\omega$-hypermultiplet $\omega = \phi^1 - \mathrm{i}\phi^2$, we observe

$$
\nabla^{++}\omega = D^{++}\omega + \mathrm{i}\sqrt{2}\mathcal{V}^{++}\omega \tag{77}
$$

hence the $U(1)$-charge of $\omega$ is $e = \sqrt{2}$. In our previous paper [28] it was shown that the effective actions of the charged complex $\omega$-hypermultiplet and the charged $q$-hypermultiplet, interacting with background $U(1)$ gauge superfield $\mathcal{V}^{++}$, are related by $\Gamma_\omega[\mathcal{V}^{++}] = 2\Gamma_q[\mathcal{V}^{++}]$ and the leading contribution to $\Gamma_q[\mathcal{V}^{++}]$ in the massive theory is given by

$$
\Gamma_q[\mathcal{V}^{++}] = \int d^4x d^4\theta \mathcal{F}(\mathcal{W}) + \text{c.c.} \qquad \mathcal{F}(\mathcal{W}) = -\frac{e^2}{64\pi^2}\mathcal{W}^2 \ln\frac{\mathcal{W}^2}{M^2} \,. \tag{78}
$$

Here $e$ is the charge of $q^+$ (it coincides with the charge of $\omega$ in the above correspondence), $M$ is the renormalization scale, and $\mathcal{W}$ the chiral superfield strength associated with $\mathcal{V}^{++}$. Since in our case $e = \sqrt{2}$, and taking into account eq. (72), we finally obtain

$$
\Gamma^{(1)}_{SU(2)}[\mathcal{V}^{++}] = \frac{1}{16\pi^2}\int d^4x d^4\theta\, \mathcal{W}^2 \ln\frac{\mathcal{W}^2}{M^2} \,. \tag{79}
$$

This is exactly Seiberg's low-energy effective action [14] found by integrating the $U(1)$ global anomaly and using the component analysis (note that Seiberg used the strength $\Psi = \sqrt{2}\mathcal{W}$).

Let us summarize the results. We have considered $N = 2$ super Yang-Mills theories in harmonic superspace and formulated the background field method for these theories. For the pure $N = 2$ SYM theory, the effective action is given by a path integral over the quantum gauge superfields and the fermionic and bosonic ghosts corresponding to $\omega$-hypermultiplets. This path integral representation allows one to carry out the perturbative loop calculations in the theory under consideration in a manifestly $N = 2$ supersymmetric and gauge invariant manner. The structure of the one-loop contributions to the effective action has been investigated and it has been shown that the whole one-loop contribution is stipulated only by the ghosts in the form of the effective action of the fermionic $\omega$-hypermultiplet coupled to the external super Yang-Mills field. This result has been applied to calculating the one-loop effective action in the $N = 4$ super Yang-Mills theory treated

as the $N = 2$ super Yang-Mills theory coupled to the $\omega$-hypermultiplet in the adjoint representation of the gauge group. Taking into account the structure of the one-loop effective action in the pure $N = 2$ SYM theory, we conclude that the one-loop effective action in the $N = 4$ SYM theory does not contain corrections depending on the $N = 2$ gauge superfield only. Finally, we have derived the well-known Seiberg's low-energy effective action in the harmonic superspace approach.

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